MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 7

Lemma 1. *(Silver) Let $\tau < \kappa$ be regular cardinals, such that $2^\tau \geq \kappa$. Suppose that T is a κ tree and $\mathbb{P}$ is τ^+-closed for some $\tau < \kappa$. Then forcing with $\mathbb{P}$ does not add new branches to T.*

Proof. Suppose otherwise. Let $\dot{b}$ be a name for a branch, forced to be such by the empty condition. Working in V, construct $\langle s_\sigma, p_\sigma \mid \sigma \in 2^{<\tau} \rangle$ by induction on the length of σ, such that:

1. Every $s_\sigma \in T, p_\sigma \in \mathbb{P}$ and $p_\sigma \Vdash s_\sigma \in \dot{b}$
2. If $\sigma_1 \subsetneq \sigma_2$, then $s_{\sigma_2} <_T s_{\sigma_1}$ and $p_{\sigma_2} \leq p_{\sigma_1}$
3. For all $\alpha < \tau$, there is some $\beta_\alpha < \kappa$, such that for every $\sigma \in 2^\alpha$, $s_\sigma \in T_{\beta_\alpha}$
4. For every σ, $s_{\sigma\frown 0}$ and $s_{\sigma\frown 1}$ are incomparable nodes.

At limit stages we use the closure of $\mathbb{P}$. More precisely, if α is limit, $\sigma \in 2^\alpha$, let p'_σ be stronger than all $p_{\sigma\restriction i}$ for $i < \alpha$. Also let $\beta_\alpha = \sup_{i<\alpha} \beta_i$. Then let $p_\sigma \leq p'_\sigma$ and $s_\sigma \in T_{\beta_\alpha}$ be such that $p_\sigma \Vdash s_\sigma \in \dot{b}$. We can find these since $\dot{b}$ is forced to meet every level.

For the successor stage, suppose that we have constructed p_σ, s_σ and β_α, where $\sigma \in 2^\alpha$. Using the splitting lemma, since $\dot{b}$ is a new branch, we have that there are conditions $q_{\sigma\frown 0}, q_{\sigma\frown 1}$ stronger than p_σ and nodes $s_{\sigma\frown 0}, s_{\sigma\frown 1}$, in $T_{\beta_\alpha+1}$ such that $q_{\sigma\frown 0} \Vdash s_{\sigma\frown 0} \in \dot{b}$ and $q_{\sigma\frown 1} \Vdash s_{\sigma\frown 1} \in \dot{b}$.

Now for every $f \in 2^\tau$, let p_f be stronger than all $p_{f\restriction\alpha}$, for $\alpha < \tau$. Here we use that $\mathbb{P}$ is τ^+-closed, i.e. sequences of length τ have a lower bound. Let $\beta = \sup_{\alpha<\tau} \beta_\alpha < \kappa$. For every $f \in 2^\tau$, let $q_f \leq p_f$ and $s_f \in T_\beta$ be such that $q_f \Vdash s_f \in \dot{b}$. Again here we use that $\dot{b}$ is forced to meet every level (since it is forced to be a branch).

But then by the splitting, we have that whenever $f \neq g$, $s_f \neq s_g$. But $|T_\beta| < \kappa$ and $2^\tau \geq \kappa$. Contradiction.

□

Corollary 2. *Suppose that T is an ω_2-tree, $\mathbb{Q}$ is ω_1-closed, and $2^\omega = \omega_2$. Then $\mathbb{Q}$ does not add new branches through T.*

Let G be $\mathbb{M}$-generic over V. We have to show the tree property in $V[G]$. Suppose that T is a $\aleph_2$-tree in $V[G]$. Note that since $\kappa = \aleph_2^{V[G]}$, this means that T is a κ-tree. We have to show that there is an unbounded branch through T.

Let $j : V \to N$ be an elementary embedding with critical point κ. Recall that we showed that $j(\mathbb{M})$ projects to $\mathbb{M}$, and so we can lift the embedding to $j : V[G] \to N[G^*]$.

Lemma 3. *There is a branch b through T in $N[G^*]$ (and so in $V[G^*]$).*

Proof. Note that in $N[G^*]$, $j(T)$ is a $j(\kappa)$-tree. Since the sizes of the levels of T are below the critical point, we can also assume that for every level $\alpha < \kappa$, $j(T_\alpha) = T_\alpha = j(T)_\alpha$.

Let $u \in j(T)_\kappa$, i.e. a node on the κ-th level of $j(T)$. Let $b = \{v \in j(T) \mid v <_{j(T)} u\}$. Since $j(T)$ is a tree, b is a well ordered set. Also, for every $v \in b$, there is some $\alpha < \kappa$, such that $v \in j(T)_\alpha = T_\alpha$. I.e. $b \subset T$. And since the order type of b is κ, it follows that b is an unbounded branch through T. □

We want to show that T has a branch in $V[G]$. So far, we have that T has a branch in the bigger model $V[G^*]$. Next we want to use branch preservation lemmas to show that forcing to get from $V[G]$ to $V[G^*]$ could not have added a new branch, i.e. that b must already exists in $V[G]$. The problem is that the forcing to get from G to G^* does not have the nice properties, like closure or Knaster-ness, that are used in the branch preservation lemmas.

To deal with that problem, recall that $\mathbb{M}$ is the projection of $\mathbb{P} \times \mathbb{Q}$, where $\mathbb{Q}$ is ω_1-closed in V and $\mathbb{P} = Add(\omega, \kappa)$. We will show that something similar is true about $j(\mathbb{M})$.

INTERLUDE ON PROJECTIONS:

Suppose that $\mathbb{R}$ and $\mathbb{R}^*$ are any two posets, such that $\mathbb{R}^*$ projects to $\mathbb{R}$. Let $\pi : \mathbb{R}^* \to \mathbb{R}$ be a projection, and suppose that H is $\mathbb{R}$-generic.

Definition 4. *In $V[H]$, we set $\mathbb{R}^*/H := \{p \in \mathbb{R}^* \mid \pi(p) \in H\}$.*

Lemma 5. *If G is $\mathbb{R}^*/H$ generic over $V[H]$, then G is $\mathbb{R}^*$-generic over V, and so $V \subset V[H] \subset V[H][G] = V[G]$.*

Proof. G is a filter by assumption, so it is enough to show genericity. Suppose that $D \in V$ is a dense subset of $\mathbb{R}^*$. Let $D^* = D \cap \mathbb{R}^*/H$. We claim that D^* is a dense subset of $\mathbb{R}^*/H$. Fix $p \in \mathbb{R}^*/H$. In V, let $D_p = \{\pi(q) \mid q \in D, q \leq p\}$.

Claim 6. *D_p is dense below $\pi(p)$.*

Proof. For any $r \in \mathbb{R}$, $r \leq \pi(p)$, using that π is a projection, let $p' \in \mathbb{R}^*$ be such that $\pi(p') \leq r$. Then let $q \leq p'$ be in D. Then $\pi(q) \in D_p$ and $\pi(q) \leq r$. □

So, let $r \in D_p \cap H$. Say $r = \pi(q)$ for some $q \in D$, with $q \leq p$. Then $q \in D^*$.

Since G is $\mathbb{R}^*/H$-generic, we have that $D^* \cap G \neq \emptyset$, and so $D \cap G \neq \emptyset$. □

Next we give an alternative definition for projections:

Definition 7. *$\mathbb{R}^*$ projects to $\mathbb{R}$ iff whenever G is $\mathbb{R}^*$-generic, then in $V[G]$, we can define a $\mathbb{R}$-generic filer.*

Definition 8. *We say that $\mathbb{R}^*$ is isomorphic to $\mathbb{R}$ if $\mathbb{R}^*$ projects to $\mathbb{R}$ and $\mathbb{R}$ projects to $\mathbb{R}^*$.*

BACK TO THE MITCHELL THEOREM:

Recall that $\mathbb{P}$ is $Add(\omega, \kappa)$ and $j : V \to N$ is an elementary embedding with critical point κ, and so $j(\mathbb{P}) = Add(\omega, j(\kappa))$. Let H be $\mathbb{P}$ generic over V. Define $\mathbb{P}^*$ to be the set of all conditions p in $j(\mathbb{P})$ such that $\mathrm{dom}(p) \cap \kappa \times \omega$ is empty. I.e. $\mathbb{P}^* = Add(\omega, j(\kappa) \setminus \kappa)$.

Lemma 9. *In $V[H]$, $\mathbb{P}^*$ is isomorphic to $j(\mathbb{P})/H = \{p \in j(\mathbb{P}) \mid p \restriction \kappa \times \omega \in H\}$.*

Proof. For the first direction, suppose that H^* is $\mathbb{P}^*$-generic over $V[H]$. In $V[H][H^*]$, define $K := \{p \in j(\mathbb{P})/H \mid p \restriction j(\kappa) \setminus \kappa \times \omega \in H^*\}$. We want to show that K is $j(\mathbb{P})/H$ generic over $V[H]$. It is a filter because both H and H^* are. For genericity, suppose that $D \in V[H]$ is a dense subset of $j(\mathbb{P})/H$. Let $D^* = \{p \restriction j(\kappa) \setminus \kappa \times \omega \mid p \in D\}$. Then D is a dense subset of $\mathbb{P}^*$, so there is some $q \in D \cap H^*$. Let p witness that q is in D^*. Then $p \in D \cap K$.

For the other direction, suppose that K is $j(\mathbb{P})/H$ generic over $V[H]$. In $V[H][K]$, define $H^* := K \cap \mathbb{P}^*$. H^* is a filter because K is a filter and for any two $p, q \in \mathbb{P}^*$, $p \cup q$ is also in $\mathbb{P}^*$. For genericity, suppose that $D \in V[H]$ is a dense subset of $\mathbb{P}^*$. Then the set $E = \{p \in j(\mathbb{P})/H \mid p \restriction j(\kappa) \setminus \kappa \times \omega \in D\}$ is a dense subset of $j(\mathbb{P})/H$. Let $p \in E \cap K$ and $q = p \restriction j(\kappa) \setminus \kappa \times \omega$. Then $q \in D \cap H^*$.

□